Differential Equations are equations containing one or more derivatives of the unknown function.

For example, $\frac{d^2y}{dx^2} + \frac{dy}{dx} = xy.$ ← 2-nd order ODE

$\frac{dy}{dx} = xy.$ ← 1-st order ODE.

Many processes in physics, medicine, chemistry can be written mathematically using ODE models

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We want to describe the following situation:
Growth rate of a population at any time is proportional to the population size at that time.

Let $P(t)$ denote the size of the population at time t (number of individuals at time t).

Then $\frac{dP}{dt} = r \, P(t)$

r ← some constant of proportionality

We will restrict ourselves to ~~the~~ 1-st order differential equations of the form

$$\frac{dy}{dx} = f(x)\,g(y)$$

The RHS is the product of two functions; one function depends on x, the other one depends on y.

Such equations are called separable dif. eq-ns

Method to solve separable dif. eq-ns

We separate the variables x and y, so that one side of eqiation depends only on y, and the other side only on x:

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

Then we integrate both sides, assuming $g(y) \neq 0$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

Example : Separable Equation

$$\frac{dz}{dt} = te^{z}$$

$z(0) = 0$ ← the initial condition

or

$z = 0$ when $t = 0$.

First we separate the variables :

$$\frac{dz}{e^z} = t\,dt$$, then we integrate

$$\int \frac{dz}{e^z} = \int e^{-z}dz = \int t\,dt$$

$$-e^{-z} = \frac{t^2}{2} + c$$

$$e^{-z} = -\frac{t^2}{2} - c$$

$$\ln e^{-z} = \ln\left(-\frac{t^2}{2} - c\right)$$

$$-z = \ln\left(-\frac{t^2}{2} - c\right)$$

$$z(t) = -\ln\left(-\frac{t^2}{2} - c\right)$$

$$z(0) = 0$$

$$0 = -\ln\left(-\frac{0^2}{2} - c\right)$$

or

$$0 = -\ln(-c)$$

$$0 = \ln(-c)$$

$-c = 1$

$\boxed{c = -1.}$

$$\boxed{z(t) = -\ln\left(-\frac{t^2}{2} + 1\right)}$$

Separable Dif. Equations

include two special cases

$\frac{dy}{dx} = f(x)$ $[g(y)=1]$	$\frac{dy}{dx} = g(y)$ $[f(x)=1]$
Pure time Dif. Eq-ns	Autonomous Dif. Eq-ns
$\frac{dy}{dt} = t^2$; $\frac{dy}{dx} = \sin x$	$\frac{dy}{dx} = y^2$; $y=y(x)$
$y=y(t)$; $y=y(x)$	$\frac{dy}{dt} = y(y-1)$; $y=y(t)$

Example

Suppose that a tree grows according to the equation

$$\frac{dH}{dt} = 7e^{-0.2t}$$

$H(0) = 50$ (cm) ← Pure-time Dif. eq.

(a) Find the general solution to the equation

$$dH = 7e^{-0.2t}\,dt.$$

$$\int dH = \int 7e^{-0.2t}\,dt$$

$$H(t) = 7\int e^{-0.2t}\,dt = \frac{7e^{-0.2t}}{-0.2} + C =$$

$$= -35e^{-0.2t} + C.$$

$$\underline{H(t) = -35e^{-0.2t} + C}$$

$$\int dH(t) = \int 7e^{-0.2t}\,dt$$

$$H(t) = 7\int e^{-0.2t}\,dt = \frac{7e^{-0.2t}}{-0.2} + C = -35e^{-0.2t} + C.$$

(b) Find the particular solution of the equation corresponding to the initial condition $H(0) = 50$ (cm)

$$H(t) = -35e^{-0.2t} + C$$

$$H(0) = -35e^{-0.2\cdot 0} + C = -35 + C = 50 \Rightarrow$$

$$\boxed{C = 85}$$

$$\boxed{H(t) = -35e^{-0.2t} + 85}$$

(c) Determine the height of the tree after 5 years ($H(5)$) and after 7 years ($H(7)$)
How much did the tree grow b/n $t=7$ and $t=5$?

$$H(5) = -35e^{-0.2\cdot 5} + 85 = -\frac{35}{e} + 85 = 72.08(cm)$$

$$H(7) = -35e^{-1.4} + 85 = 76.3(cm)$$

$$H(7) - H(5) = 4.25(cm).$$

(e) Determine how much did the tree grow between $t=5$ and $t=7$ using the FTC

$$\int_5^7 H'(t)\,dt = H(t)\Big|_{t=5}^{t=7} = \overbrace{H(7)-H(5)}$$

$$\int_5^7 7e^{-0.2t}\,dt = \frac{7e^{-0.2t}}{-0.2}\Big|_{t=5}^{t=7} =$$

$$= -35e^{-0.2t}\Big|_{t=5}^{t=7} = -35e^{-1.4} + 35e^{-1} =$$

$$= 4.247(cm) = \underbrace{H(7)-H(5)}_{\text{how much did the tree grow b/n } t=5 \text{ and } t=7}$$

The FTC

$$\boxed{\int_a^b f(t)\,dt = F(b)-F(a), \quad F'(t) = f(t)}$$

Model 1 Exponential Model of Population Growth

Let $P(t)$ denote the size of a population at time t. The Growth rate of the population is proportional to the size of the population at that time.

$$\boxed{\frac{dP}{dt} = r\,P(t)}$$

$P(0) = P_0$ ← initial size of the population

r is a constant of proportionality; it is caled per capita growth rate.

→ Separate the variables and integrate:

$$\int \frac{dP}{P} = \int r\,dt$$

$$\ln|P| = rt + C$$

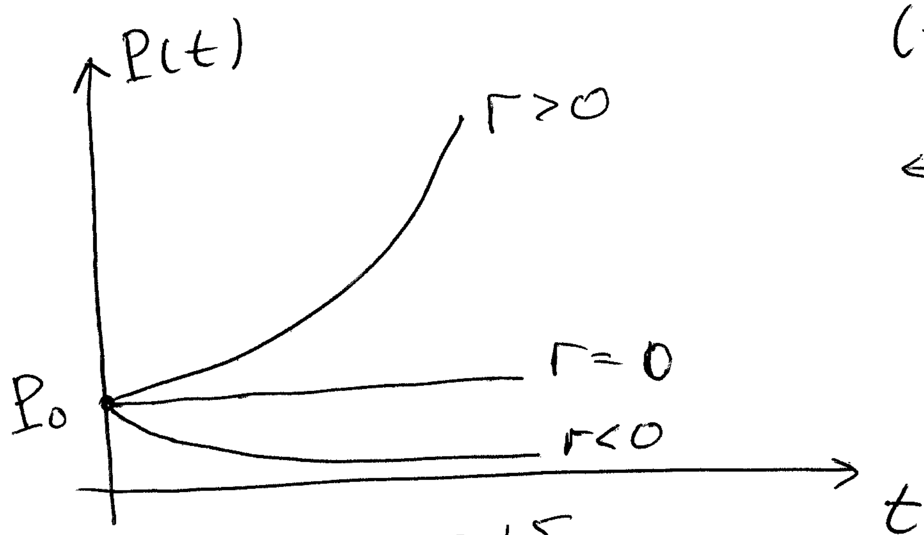
$$|P| = e^{rt+C} = e^{rt}\cdot \underbrace{e^{C}}_{=K \text{ (another constant)}} = e^{rt}\cdot K$$

$$P(t) = \pm K e^{rt} = \bar{C} e^{rt}.$$

$$P(0) = P_0 = \bar{C} e^{r\cdot 0} = \bar{C} \Rightarrow \boxed{\bar{C} = P_0}$$

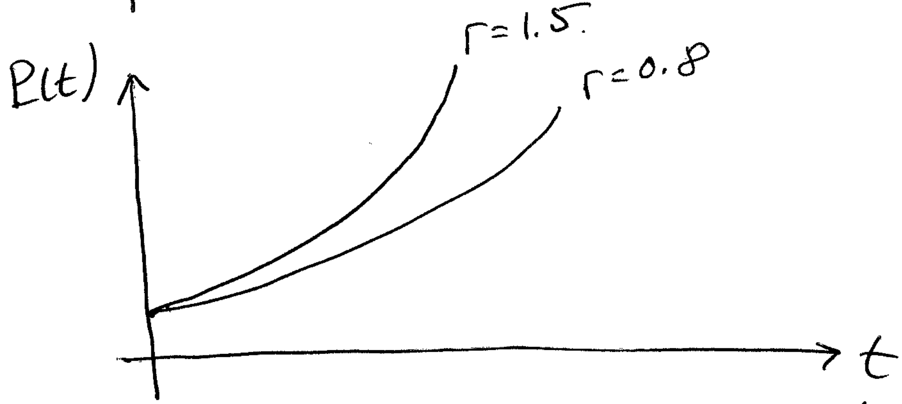
$$P(t) = \bar{C} e^{rt} = P_0 e^{rt}$$

$P = 0$ does satisfy $\frac{dP}{dt} = rP$. Thus, $\bar{C}$ can be zero, even though there is no c: $e^c = \bar{C} = 0$. (We lost this solution, because we divided by P in the first step and we assumed $P \neq 0$)



(t, P)-plane

← these are solution curves that start at $(0, P_0)$ for different r's.



* If $r<0$ $\frac{dP}{dt} = rP < 0$ and the population decreases exponentially (dies out)

→ If $r>0$, the model predicts unlimited growth (it is not realistic). Since in this case, the population grows without a bound. This can happen if individual from the population are not limited by food availability or competition

For example,

If we start a bacteria colony on a nutrient-rich substrate, then the bacteria start growing. When the substrate becomes more crowded and the food source depleted, the growth will be restricted, and a different model will be required to describe the situation

$\boxed{\text{Model 2}}$ Logistic Differential Equation

We had (Model 1) $\frac{dP}{dt} = \underline{\underline{r}}\,P$

constant

Now, we assume that $\frac{dP}{dt} = \underbrace{h(P)}\cdot P$

We replaced r by a function $h(P)$, the function that depends on the population size. We want to ~~pic~~ choose $h(P)$ so that

1. $h(P) \approx r$, when the population P is small
2. $h(P)$ decreases as P increases
3. $h(P)$ is negative, when P is sufficiently large.

The simplest function having these properties is $h(P) = r - aP$

a positive constant

$$\frac{dP}{dt} = \underbrace{(r - aP)}_{h(P)} \cdot P = r\left(1 - \frac{a}{r} P\right) \cdot P =$$

$$\frac{a}{r} = \frac{1}{K}.$$

$$= r\left(1 - \frac{P}{K}\right) \cdot P \qquad \frac{1}{K} = \frac{a}{r} \text{ or}$$

$$\boxed{K = \frac{r}{a}}$$ – is called the carrying capacity

The model was introduced by Belgian mathematician Verhulst in 1838.

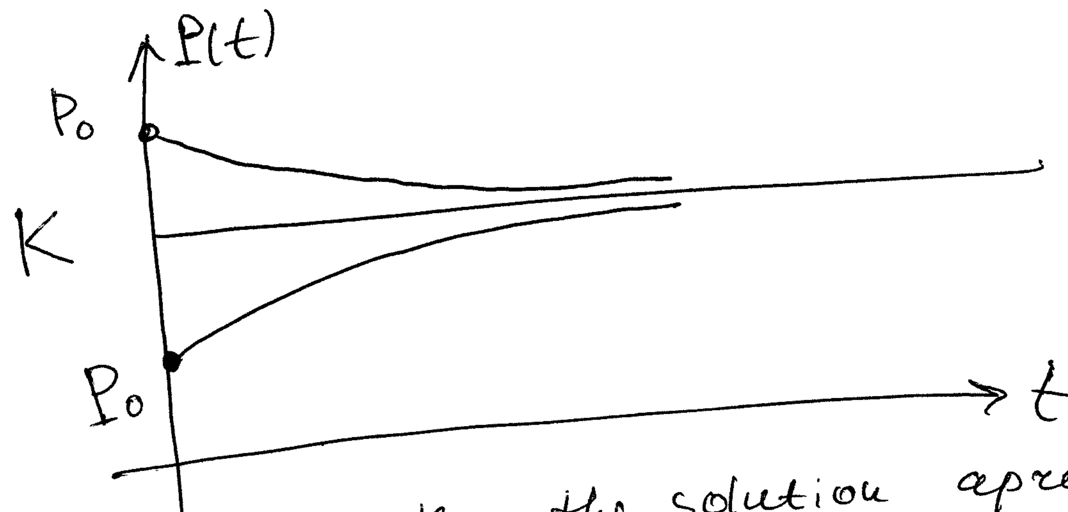
$$\boxed{\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)}$$

r, K are constants.

If $P(0) = P_0$ ← the initial size of the population

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}}$$

$(t, P(t))$-plane



→ If $P_0 > ~~B~~ K$. the solution aproach K in the long term (the solution curve decreases)

→ If $P_0 < K$, the solution curve increases and aproach K as well

If $P_0 = K$ then we will stay at $P(t) = K$.

The Allee Effect Model

$\boxed{\text{Model 3}}$ A simple extension of the logistic eq-n, which takes into consideration the fact that some populations may experience drop of per capita growth rate when the population density falls below the certain level:

$$\frac{dP}{dt} = rP(P-a)\left(1-\frac{P}{K}\right)$$

In other words, if the population falls to a certain minimum, it will keep decreasing and eventually become extinct.

a, r, K are positive constants

the $(t, P(t))$ plane